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TWO EXAMPLES OF LOCAL ERGODIC DIVERGENCE

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In memory of our friend and colleague Shlomo Horowitz

ABSTRACT

This note contains the first example of a 1-parameter semigroup $\{T_t, t \geqq 0\}$ of linear contractions in L_p $(1 < p < \infty)$ for which the assertion of the local ergodic theorem ($t^{-1}\int_0^t T_s f ds$ conv. a.e. as $t \to 0+0$ for all $f \in L_p$) fails to be true. The first example is a continuous semigroup of unitary operators in L_2, the second a power-bounded continuous semigroup of positive operators in L_1. This answers problems of Kubokawa, Fong and Sucheston.

In this note we show that the local ergodic theorem fails to hold for continuous groups of unitary operators in L_2 if no assumption of positivity is added. This answers in the negative a question raised by Y. Kubokawa [9].

Our second example shows that the local ergodic theorem does not hold in general for continuous semigroups $\{T_t, t \geqq 0\}$ of positive bounded linear operators in L_1 with $\sup_t \|T_t\| < \infty$. After submitting the paper we learned that another example of such a semigroup has recently been given by R. Sato [12]. We think that our example remains to be of some interest since it is more direct and does not depend on a rather non-trivial theorem of Derriennic and Lin [5].

Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space and $\{T_t, t \geqq 0\}$ a semigroup of bounded linear operators in L_p $(1 \leqq p < \infty)$. The operators T_t are called positive if $f \geqq 0$ implies $T_t f \geqq 0$, they are called contractions if $\|T_t\| \leqq 1$. The semigroup-property means that $T_{t+s} = T_t \circ T_s$ $(t, s \geqq 0)$. The semigroup is called continuous if $\lim_{t \to s} \|T_t f - T_s f\|_p = 0$ holds for all $f \in L_p$ and for all $s \geqq 0$ (or, equivalently, for $s = 0$). It is called continuous at $\{s > 0\}$ if the condition $\lim_{t \to s} \|T_t f - T_s f\|_p = 0$ holds for all $f \in L_p$ and all $s > 0$. In this case the integrals

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$$S_t f = \int_0^t T_s f ds$$

can be defined either by using approximating Riemann sums or by choosing suitable representatives of the equivalence classes $T_s f$. Local ergodic theorems assert that

$$\lim_{t \to 0+0} t^{-1} S_t f \text{ exists a.e. for all } f \in L_p \tag{1}$$

under various conditions. Krengel [7] and Ornstein [11] proved (1) for continuous semigroups of positive contractions in L_1. This has been substantially generalized in various directions notably by Akcoglu, Chacon, Ornstein, Terrell, Kipnis, Kubokawa, Fong, Sucheston, Baxter, McGrath, and R. Sato. Recently Akcoglu–Krengel [1], [2] have obtained convergence theorems in a more general setting than that of indefinite integrals by studying additive processes, i.e. families $\{F_t, t > 0\}$ with $F_{t+s} = F_t + T_t F_s$ instead of $S_t f$. (This contains e.g. the Lebesgue-differentiation theorems for functions of bounded variation.)

EXAMPLE 1. Kubokawa [9] proved (1) for continuous semigroups of positive bounded linear operators in L_p $(1 \leqq p < \infty)$ with $T_0 =$ identity. He mentioned that it was one of the most interesting open problems on ergodic theorems, whether the condition of positivity could by dropped. We shall now show that the answer is negative for $p = 2$.

Let L_2 be the complex L_2-space associated with the unit interval Ω together with the Lebesgue measure μ. For each $t \in \mathbf{R}$ we will define a unitary operator $T_t : L_2 \to L_2$ such that

(i) for each t, s $T_t \circ T_s = T_{t+s}$, $T_0 =$ identity $= I$, and $\lim_{s \to t} \| T_s f - T_t f \|_2 = 0$ for all $f \in L_2$ and all t, and also such that

(ii) there exists a function $f_0 \in L_2$ for which $t^{-1} S_t f_0$ diverges a.e.

If $\{\varphi_n\}_{n=1}^{\infty}$ is any complete orthonormal basis for L_2 and $\{\lambda_n\}_{n=1}^{\infty}$ an arbitrary collection of non-zero real numbers, then

$$T_t \left(\sum_{n=1}^{\infty} \alpha_n \varphi_n \right) = \sum_{n=1}^{\infty} e^{i\lambda_n t} \alpha_n \varphi_n$$

defines unitary operators T_t satisfying the conditions in (i). Here the α_n's are complex numbers with $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$ and $\sum_{n=1}^{\infty} \alpha_n \varphi_n$ represents an arbitrary function f in L_2.

For each $k \geqq 1$ let $P_k : L_2 \to L_2$ be the projection $\sum_{n=1}^{\infty} \alpha_n \varphi_n \to \sum_{n=1}^{k} \alpha_n \varphi_n$. It is easy to see that

$$S_t f = \sum_{n=1}^{\infty} \frac{1}{i\lambda_n} (e^{i\lambda_n t} - 1) \alpha_n \varphi_n \qquad (t > 0).$$

Below we shall make use of the fact that

$$\left| \frac{e^{i\lambda_n t} - 1}{i\lambda_n t} \right| \leqq 1 \quad \text{and} \quad \frac{e^{i\lambda_n t} - 1}{i\lambda_n t} \to 1$$

as $t \to 0+0$. This also implies that $t^{-1}S_t f \to f$ in L_2 as $t \to 0+0$.

LEMMA. *Given any sequence $(\varepsilon_k)_{k=1}^{\infty}$ with $\varepsilon_k > 0$ there exist sequences $(\lambda_n)_{n=1}^{\infty}$ and $(t_k)_{k=1}^{\infty}$ with $\lambda_n > 0$ and $0 < t_k \to 0$ such that $\| P_k - t_k^{-1} S_{t_k} \| < \varepsilon_k$ $(k \geqq 1)$.*

PROOF. Choose $\lambda_1 > 0$ arbitrarily. Then choose $0 < t_1 < 1$ so small that

$$\left| \frac{e^{i\lambda_1 t_1} - 1}{i\lambda_1 t_1} - 1 \right| < \varepsilon_1.$$

If $\lambda_1, \cdots, \lambda_k$ and $t_1, \cdots, t_k$ are already chosen, choose $\lambda_{k+1} > 0$ sufficiently large so that

$$\left| \frac{e^{i\lambda_{k+1} t_m} - 1}{i\lambda_{k+1} t_m} \right| < \varepsilon_m$$

for each $m = 1, 2, \cdots, k$. Then choose $0 < t_{k+1} < (k+1)^{-1}$ so small that

$$\left| \frac{e^{i\lambda_n t_{k+1}} - 1}{i\lambda_n t_{k+1}} - 1 \right| < \varepsilon_{k+1}$$

for all $n \leqq k+1$. □

By Menchoff's example [10] we know that there is an orthonormal basis $\{\varphi_n\}$ and a vector $f_0 \in L_2$ such that $P_k f_0$ diverges a.e. as $k \to \infty$. Choosing the ε_k's sufficiently small and determining the λ_n's and t_k's as in the lemma we see that $t_k^{-1} S_{t_k} f_0$ also diverges a.e.

Burkholder [4] has already used Menchoff's example to obtain a counterexample to a.e. convergence of $n^{-1} \Sigma_{k=0}^{n-1} T^k f_0$ for a contraction T in L_2. Akcoglu and Sucheston [3] have extended this to unitary operators. Our example is, of course, inspired by these ideas.

EXAMPLE 2. In [8] Kubokawa proved (1) for continuous semigroups of positive bounded operators in L_1 assuming $T_0 =$ identity. He quoted the result of Krengel–Ornstein and some of the authors mentioned in the introduction saying that they had proved (1) under "little different conditions" (from those of Krengel–Ornstein).

As he didn't mention that those "little different conditions" required substantially different proofs it may be of interest that the result of Kubokawa does not hold under conditions which are even much less different, namely, for general

T_0. We now construct a σ-finite measure space $(\Omega, \mathcal{F}, \mu)$, a continuous semigroup $\{T_t, t \geqq 0\}$ of positive bounded operators in L_1, and an $f_0 \in L_1^+$ for which $t^{-1}S_t f_0$ diverges on a set of positive measure.

Let $\Omega_i = \{(i, x) : x \in \mathbf{R}\}$ $(i = 1, 2, \cdots)$, $\Omega_0 = \{(0, x) : x \in [0, 1]\}$, and $\Omega = \bigcup_{i=0}^{\infty} \Omega_i$. μ restricted to Ω_i shall be the Lebesgue-measure.

Let $\beta_n = n^{-1}$ $(n \geqq 1)$. Let $B_1, B_2, \cdots$ be a sequence of measurable subjets of Ω_0 with $\mu(B_n) = \beta_n$ such that each $\omega_0 \in \Omega_0$ belongs to infinitely many B_n (e.g. the B_n's are "independent" under the restriction of μ to Ω_0). Let $\alpha_n = \beta_n - \beta_{n+1}$, $c_n = 2^{-n}(n!)^{-1}$, and $\varepsilon_n = 2^{-2}c_n$.

Before proceeding with the formal argument we sketch the heuristic idea: f_0 is a mass distribution putting mass α_i in Ω_i $(i \geqq 1)$ and this mass shall be uniformly distributed in $\{i\} \times [-\varepsilon_i, 0]$. T_t transports this mass with unit speed to the right. On Ω_i we shall specify finitely many disjoint intervals $B_{ik} = \{i\} \times [c_k \cdot 2c_k]$ such that $k \leqq i \leqq 2k - 1$. When some mass arrives in one of these B_{ik} it moves on and simultaneously the same amount is mapped to B_k. Any mass in Ω_0 is immediately killed. The mass put into B_k shall be distributed there with a uniform density. In any moment when the whole interval $[-\varepsilon_i, 0]$ has moved into B_{ik} the total mass α_i is sent to B_k. This happens during most of the time interval $[c_k, 2c_k]$. As B_k receives mass from all the Ω_i with $k \leqq i \leqq 2k - 1$ the total mass sent there is $\alpha_k + \alpha_{k+1} + \cdots + \alpha_{2k-1} = \beta_k/2$. Thus during most of $[c_k, 2c_k]$ $T_t f_0$ has density $\frac{1}{2}$ in B_k. This suffices to show that $\limsup_{t \to 0} t^{-1}S_1 f_0 \geqq 1/8$ in Ω_0. On the other hand, between time $2c_k + \varepsilon_i \leqq 3c_k$ and time c_{k-1} no mass is sent to Ω_0. If k is large $3c_k$ is much smaller than c_{k-1} and we get $\liminf_{t \to 0} t^{-1}S_t f_0 = 0$ on Ω_0.

We now formalize the definitions: Let

$$f_0(i, x) = \begin{cases} 0 & \text{if } i = 0 \text{ or } i \geqq 1 \text{ and } x \notin [-\varepsilon_i, 0], \\ \alpha_i \varepsilon_i^{-1} & \text{if } i \geqq 1 \text{ and } x \in [-\varepsilon_i, 0]. \end{cases}$$

Let $K := \{(i, k) : 1 \leqq k \leqq i \leqq 2k - 1\}$ and $K(i) := \{k : (i, k) \in K\}$. The semigroup $\{T_t, t \geqq 0\}$ is defined as follows: For any $g \in L_1$ with $\{g \neq 0\} \subset \Omega_0$ put $T_t g = 0$ $(t \geqq 0)$. For any $g \in L_1$ with $\{g \neq 0\} \subset \Omega_i$ $(i \geqq 1)$ let

$$(T_t g)(j, x) = \begin{cases} g(j, x - t) & \text{if } j \geqq 1, \\ \displaystyle\sum_{k \in K(i)} 1_{B_k(j,x)} \mu(B_k)^{-1} \int_{B_{i,k}} g(i, x - t)dx & \text{if } j = 0. \end{cases}$$

It follows from the disjointness of the intervals $B_{i,k}$ ($k \in K(i)$) that $\| T_t \| = 2$ for all $t \geqq 0$. It is not hard to see that $\{T_t, t \geqq 0\}$ is a continuous semigroup of positive linear operators.

Note that

$$\alpha(i, k, t) := \int_{B_{ik}} f_0(i, x - t)dx$$

can be positive only for $t \in [c_k, 2c_k + \varepsilon_i] \subset [c_k, 3c_k]$ and it is bounded by α_i. Moreover $\alpha(i, k, t) = \alpha_i$ for $t \in [c_k + \varepsilon_i, 2c_k] \supset [\frac{3}{2}c_k, 2c_k]$. As $2^{-1}\beta_k = \Sigma_{i=k}^{2k-1} \alpha_i$ and the intervals $[c_k, 3c_k]$ are disjoint for $k \geqq 2$ it follows that $(T_t f_0)(0, \cdot)$ is $\leqq 2^{-1}$ for all $t \leqq c_2$. For $t \in [3 \cdot 2^{-1} c_k, 2c_k] T_t f_0 \geqq 2^{-1} 1_{B_k}$ and for $t \in [3c_k, c_{k-1}]$ $(T_t f_0)(0, \cdot) = 0$. This suffices to prove that $t^{-1} S_t f_0$ diverges in Ω_0 as $t \to 0 + 0$, because $c_{k-1}/c_k \to \infty$.

□

Of course T_0 in this example is a very peculiar operator. In our forthcoming paper [2] the result of Kubokawa is extended to positive additive processes $\{F_t, t > 0\}$ and semigroups $\{T_t, t > 0\}$ of positive bounded operators in L_p, continuous at $\{s > 0\}$, under a rather weak condition on the "initially conservative part".

Note, that the semigroup in Example 2 is bounded, i.e. there exists a $K < \infty$ with $\sup\{\| T_t \|, 0 < t < \infty\} \leqq K$. Fong and Sucheston [6] had already proved (1) for such semigroups and $p = 1$ on a part Y of the space and they mentioned that the local behavior on the complement Z was not easy to ascertain by their methods. Example 2 shows that one may, in fact, have divergence on Z.

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